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Stable circular orbits of freely moving balls on rotating discs

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If balls are set on rotating discs these balls do not leave the rotating plane but move on circles. The frequency of this circular movement depends on neither radius nor mass of the balls, nor on the radius of the circle. It depends only on the frequency of the rotating disc. It can be shown that this surprising phenomenon, which can easily be observed, is due to friction between ball and disc. A theoretical solution is given.

I. PHENOMENON

A surprising phenomenon may be observed while experimenting with balls on rotating discs to demonstrate Coriolis acceleration. The rotating disc must be accurately horizontal. A ball may rest in the center of the disc. The ball is given a push. To show effects of rotating reference systems this push is usually a strong one. The ball moves outward and leaves the disc.

If the disc rotates relatively fast, say with one rotation per second, and if the push is not very strong, an unexpected phenomenon occurs. The ball runs on a circular orbit back to the center.[1] The radius of the orbit depends on the initial impulse. The frequency of the orbital motion depends only on the frequency of the rotating disc. It is given by

$$\boldsymbol{\omega}_c = (2/7)\,\boldsymbol{\omega}_d, \tag{1}$$

where $\boldsymbol{\omega}_c$ = frequency of orbital motion and $\boldsymbol{\omega}_d$ = rotational frequency of disc. The sense of rotation is the same for $\boldsymbol{\omega}_c$ and $\boldsymbol{\omega}_d$.

During the movement the ball rotates because of the friction between ball and disc. Figure 1 shows the circular orbit of the center of the ball.[2]

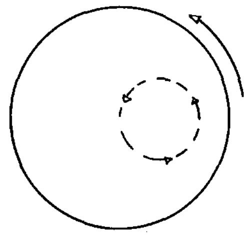

Fig. 1. Orbital path of a freely moving ball on a rotating disc.

The phenomenon is not restricted to the special case of motion starting at the center of the disc. If balls are set on the rotating disc at any place, the results are circular orbits. The center of the orbit depends on the starting point and initial impulse. The rotational frequency still is given by (1). It does not even depend on the radius of the ball. Small and big balls run with the same frequency of rotation on circular orbits of any radius. With hollow balls (ping-pong balls) the phenomenon is the same but with a changed rotational frequency,

$$\boldsymbol{\omega}_c = (2/5)\boldsymbol{\omega}_d. \tag{2}$$

If the plane of the rotating disc deviates from the horizontal adjustment, a drift movement is superimposed.

In Fig. 2 the disc may be inclined along the y axis. If the disc does not rotate, the ball is accelerated in the y direction. On a rotating disc a constant drift velocity of the center of the circular orbit results in the x direction. The balls move on orbits which drift in a direction perpendicular to the direction of inclination.

II. THEORY

A. Basic Idea

The following considerations refer to the inertial reference system. The phenomenon is caused by the friction between ball and disc. The basic assumption is that the ball rolls without slipping. In this case velocity of the ball surface $\mathbf{v}_b$ and the velocity of the rotating disc $\mathbf{v}_d$ are equal for the point of contact (see Fig. 3). The velocity of the rotating disc at the point of contact is given by

$$\mathbf{v}_d = \boldsymbol{\omega}_d \times \mathbf{r}.$$

The velocity of the surface of the ball at the point of contact is the sum of the velocity of the center of the ball and the velocity resulting from the spin of the ball:

$$\mathbf{v}_b = \dot{\mathbf{r}}_b + \boldsymbol{\omega}_b \times \mathbf{R}.$$

Since $\mathbf{v}_b = \mathbf{v}_d$, the angular velocity of the ball $\boldsymbol{\omega}_b$ may be deduced in terms of $\mathbf{r}$ and $\dot{\mathbf{r}}$. With the angular velocity $\boldsymbol{\omega}_b$ the angular momentum $\mathbf{L}$ is determined. If the ball moves, $\mathbf{L}$ changes. This change of angular momentum $\mathbf{L}$ must be caused by a torque, caused by friction force at the point of contact. Simultaneously this force changes the translational momentum $\mathbf{P}$ of the ball. To get the right change of $\mathbf{L}$ the frictional force must be perpendicular to $\dot{\mathbf{r}}$. Thus orbital movements arise. By considering a special point of the orbit it may be made plausible that the frictional force $\mathbf{F}$ is perpendicular to $\dot{\mathbf{r}}$.

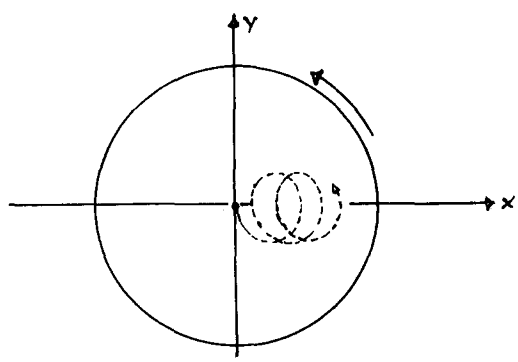

Fig. 2. Path of a ball moving on a rotating disc inclined along the y axis.

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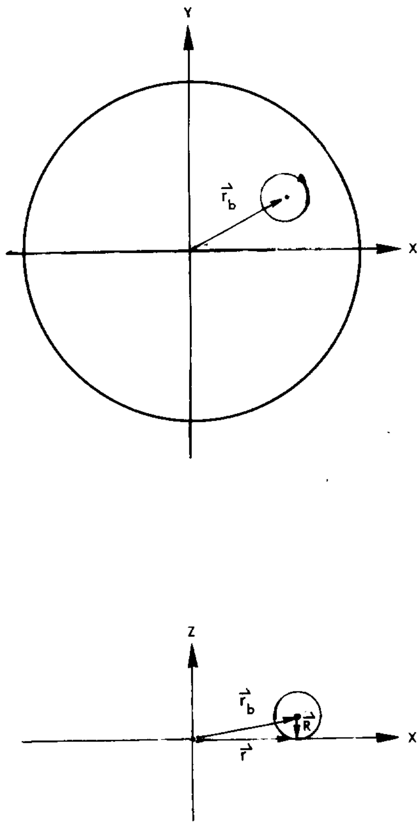

Fig. 3. Denotions of position.

Look at Fig. 4. Two points nearly opposite to the center of rotation of the disc are drawn. The ball moves on the rotating disc from position ① to position ②. In this special position the directions of $\Delta\mathbf{L}$ and $\dot{\mathbf{r}}$ coincide. The difference of the angular momentum $\Delta\mathbf{L}$ is the effect of the torque by the frictional force $\mathbf{F}$. By this, $\mathbf{F}$ must be perpendicular to $\Delta\mathbf{L}$ and $\dot{\mathbf{r}}$.

This consideration cannot be applied at a general position on the circular orbit. Furthermore, it does not take into account that the angular momentum of the ball is given by the rotation of the disc *and* $\dot{\mathbf{r}}$. To get a satisfying solution we must solve the equations of motion.

B. Solution

1. *Equations of motion*

Velocity of any point on the disc is

$$\mathbf{v}_d = \boldsymbol{\omega}_d \times \mathbf{r}. \tag{3}$$

Velocity of the surface of the ball at the point of contact is

$$\mathbf{v}_b = \dot{\mathbf{r}} + \boldsymbol{\omega}_b \times \mathbf{R}. \tag{4}$$

Since we assume no slip at the point of contact,

$$\mathbf{v}_b = \mathbf{v}_d. \tag{5}$$

Inserting (3) and (4) in (5), we get

$$\boldsymbol{\omega}_b \times \mathbf{R} + \dot{\mathbf{r}} = \boldsymbol{\omega}_d \times \mathbf{r}. \tag{6}$$

We rearrange (6):

$$\boldsymbol{\omega}_b \times \mathbf{R} = \boldsymbol{\omega}_d \times \mathbf{r} - \dot{\mathbf{r}}. \tag{7}$$

We differentiate the equation:

$$\dot{\boldsymbol{\omega}}_b \times \mathbf{R} = \boldsymbol{\omega}_d \times \dot{\mathbf{r}} - \ddot{\mathbf{r}}. \tag{8}$$

The derivative of angular momentum $\dot{\mathbf{L}}$ equals the torque of the friction force $\mathbf{F}$ with respect to the center of mass:

$$\dot{\mathbf{L}} = I\,\dot{\boldsymbol{\omega}}_b = \mathbf{R} \times \mathbf{F}. \tag{9}$$

I denotes the moment of inertia of the ball. $\mathbf{F}$ accelerates the ball as a whole:

$$\mathbf{F} = m \cdot \ddot{\mathbf{r}}. \tag{10}$$

We insert (10) into (9):

$$I\,\dot{\boldsymbol{\omega}}_b = \mathbf{R} \times m \cdot \ddot{\mathbf{r}}. \tag{11}$$

Now we combine (11) and (8):

$$[(\mathbf{R}/I) \times m \cdot \ddot{\mathbf{r}}] \times \mathbf{R} = \boldsymbol{\omega}_d \times \dot{\mathbf{r}} - \ddot{\mathbf{r}}. \tag{12}$$

The left-hand side of the equation may now be simplified:

$$(R^2/I) \cdot m \cdot \ddot{\mathbf{r}} = \boldsymbol{\omega}_d \times \dot{\mathbf{r}} - \ddot{\mathbf{r}}. \tag{13}$$

We rearrange and get the equation of motion:

$$\ddot{\mathbf{r}} = \boldsymbol{\omega}_d \times \dot{\mathbf{r}}/((\mathbf{R}^2/\mathbf{I}) \cdot \mathbf{m} + 1). \tag{14}$$

2. *Solutions of the equation of motion*

The disc rotates around the origin of the coordinate system. We now show that circular orbits of the center of the ball are solutions of Eq. (14). The ball moves on a circular path with radius $|\rho|$ and frequency ω_c. The center of the orbit $\mathbf{r}_0$ is arbitrary. (See Fig. 5.)

$$\mathbf{r} = \mathbf{r}_0 + \boldsymbol{\rho} \tag{15}$$

$$\dot{\mathbf{r}} = \boldsymbol{\omega}_c \times \boldsymbol{\rho} \tag{16}$$

$$\ddot{\mathbf{r}} = -\,\omega_c^2 \boldsymbol{\rho} \tag{17}$$

We insert (16) and (17) in (14) and get

$$-\omega_c^2 \boldsymbol{\rho} = \frac{\boldsymbol{\omega}_d \times (\boldsymbol{\omega}_c \times \boldsymbol{\rho})}{(R/I) \cdot m + 1} = \frac{-\boldsymbol{\omega}_d \cdot \omega_c \cdot \boldsymbol{\rho}}{(R/I) \cdot m + 1}, \tag{18}$$

$$\omega_c = \frac{\omega_d}{(R/I) \cdot m + 1}. \tag{19}$$

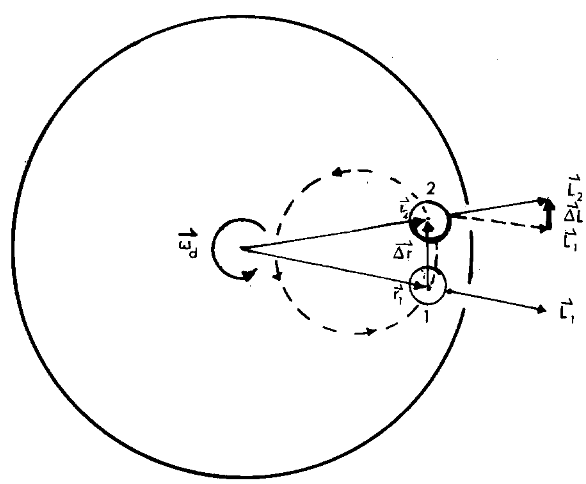

Fig. 4. Angular momentum of ball in two positions.

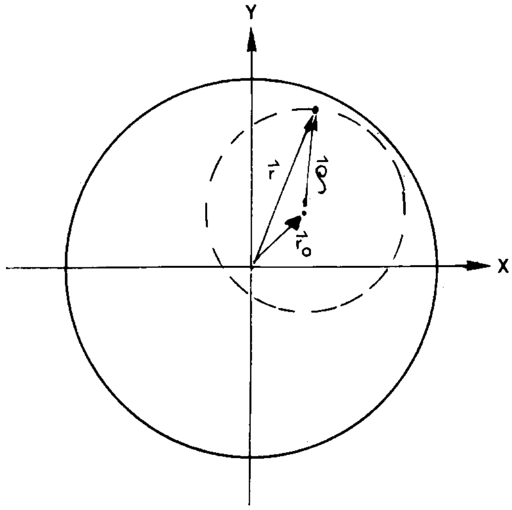

Fig. 5. Circular path of ball with arbitrary center and radius.

The angular frequency of the orbital motion is independent of the center and the radius of the orbit. This is exactly the phenomenon which is observed. The moment of inertia of a solid ball is

$$I = (2/5) \cdot R^2 \cdot m. \qquad (20)$$

We thus obtain

$$\omega_c = \omega_d \cdot (2/7). \qquad (21)$$

With a hollow ball we have

$$\omega_c = \omega_d \cdot (2/5) \qquad (22)$$

If we wish, we may find the two constants $\mathbf{r}_0$ and $|\rho|$ in terms of the initial data $\mathbf{r}(0)$ and $\mathbf{v}(0)$:

$$\dot{\mathbf{r}}(o)^2 = v^2(o) = \omega_c^2 \cdot \rho^2 \qquad (23)$$

$$\rho = v(o)/\omega_c. \qquad (24)$$

The radius of the orbit increases with increasing initial velocity. If $v(\mathrm{o}) = 0$ the spinning ball remains at any point of the disc. This is easy to observe.

We get $\mathbf{r}(\mathrm{o})$ from (15) and (16),

$$\mathbf{r}_o = \mathbf{r}(o) - \boldsymbol{\rho}(o). \qquad (25)$$

Since

$$\boldsymbol{\rho}(o) = \mathbf{v}(o) \times \boldsymbol{\omega}_c/\omega_c^2, \qquad (26)$$

$$r_o = \begin{Bmatrix} x(o) - (v_y(o)/\omega_c) \\ y(o) + (v_x(o)/\omega_c) \\ 0 \end{Bmatrix} \cdot \qquad (27)$$

3. *Motion on an inclined rotating disc*

The horizontal adjustment of the rotating disc is often inaccurate. In this case we observe that the center of the orbit drifts in a certain direction. The velocity of this drift is proportional to the inclination of the disc. The drift results from a constant force acting on the ball.

We denote the constant force $\mathbf{F}_o$. Accordingly, Eq. (11) changes to

$$m \cdot \ddot{\mathbf{r}} = \mathbf{F} + \mathbf{F}_o. \qquad (28)$$

Equation (12) now becomes

$$I \cdot \boldsymbol{\omega}_b = \mathbf{R} \times (m \cdot \ddot{\mathbf{r}} - \mathbf{F}_o). \qquad (29)$$

All equations up to (14) are modified accordingly and (14) becomes

$$(R^2/I) \cdot m \cdot \ddot{\mathbf{r}} - (R^2/I)\mathbf{F}_o = \omega_d \times \dot{\mathbf{r}} - \ddot{\mathbf{r}}. \qquad (30)$$

The solution of this equation is a superposition of the orbital movement and a constant drift velocity $\mathbf{v}_\mathrm{d}$:

$$\mathbf{r} = \mathbf{r}_o + \boldsymbol{\rho} + \mathbf{v}_d \cdot t. \qquad (31)$$

Inserting the derivatives of (31) into (30), we obtain

$$-(R^2/I) \cdot \mathbf{F}_0 = \boldsymbol{\omega}_d \times \mathbf{v}_d \qquad (32)$$

$$\mathbf{v}_d = -\mathbf{F}_o \times \boldsymbol{\omega}_d(R^2/I\omega_d^2). \qquad (33)$$

The drift velocity is thus perpendicular to $\mathbf{F}_0$.

[1] Mr. Straub, a member of our institute, built a rotating disc to demonstrate Coriolis accelerations and observed the phenomenon in 1977. Since it is easily observed and is an interesting problem of classical mechanics, I assume that it has previously been described, but as far as I know the literature, examples are given below, I have not found the problem mentioned. G. R. Fowles, *Analytical Mechanics* (Rinehart and Winston, New York, 1977); G. Hamel, *Theoretische Mechanik* (Springer, Berlin, Göttingen, and Heidelberg, 1949); G. L. Kotkin, and V. G. Serbo, *Collection of Problems in Classical Mechanics* (Pergamon, New York, 1971); I. Mestscherki, *Aufgabensammlung zur Mechanik* (Deutscher Verlag der Wissenschaften, Berlin, 1955); K. R. Symon, *Mechanics* (Addison Wesley, Reading, Mass., 1971).

[2] The effect may even be observed with a ping-pong ball on a phonograph turntable, but in this case the frequency is a bit low, and the stability of the orbit is thus poor.