Text S2. Newtonian and relativistic free-fall motion.

Here we consider the radial motion of a particle of mass m due to the gravitational field of a uniform sphere of mass M and radius R.

In the Newtonian framework, the change in gravitational potential energy of the particle from an initial position r_0 to a final position r is given by

$$\Delta U = U(r) - U(r_0) = -GMm\left[\frac{1}{r} - \frac{1}{r_0}\right]. \quad \text{(B1)}$$

If the distance travelled by the particle is small compared to r_0, that is, $|r - r_0|/r_0 << 1$, then $1/r$ is essentially given by

$$\frac{1}{r} = \frac{1}{r_0}\left(1 + \frac{r - r_0}{r_0}\right)^{-1} \approx \frac{1}{r_0}\left(1 - \frac{r - r_0}{r_0}\right) = \frac{1}{r_0} - \frac{r - r_0}{r_0^2}, \quad \text{(B2)}$$

since higher-order terms involving $(r - r_0)/r_0$ are negligible. If the particle is initially near the surface of the sphere, that is, $r_0 \approx R$, then

$$GM/r_0^2 \approx GM/R^2 = g. \quad \text{(B3)}$$

Substituting Eqs. (B2) and (B3) into Eq. (B1) reduces Eq. (B1) to approximately the change in gravitational potential energy of a particle in a uniform gravitational field

$$\Delta U \approx mgr - mgr_0.. \quad \text{(B4)}$$

The Newtonian position and velocity of the particle at time t are therefore given by the well-known equations:

$$r - r_0 = v_0(t - t_0) - \frac{1}{2}g(t - t_0)^2, \quad \text{(B5)}$$

$$v = v_0 - g(t - t_0). \tag{B6}$$

In the special-relativistic framework, if $|r - r_0|/r_0 << 1$ and $r_0 \approx R$, Eqs. (B2) and (B3) reduce the change in gravitational potential energy of the particle to

$$\Delta U = -\frac{GMm}{\sqrt{1-(v/c)^2}}\left[\frac{1}{r}-\frac{1}{r_0}\right] \approx \frac{mgr - mgr_0}{\sqrt{1-(v/c)^2}}. \tag{B7}$$

Solution of the special-relativistic equation of motion with the force derived from the gravitational potential energy $U(r)$ in Eq. (B7) yields [1-3]

$$r - r_0 = -(c^2/g)\ln\left\{\frac{1}{2}\left[\left(1+\frac{v_0}{c}\right)e^{-g(t-t_0)/c} + \left(1-\frac{v_0}{c}\right)e^{g(t-t_0)/c}\right]\right\}, \tag{B8}$$

$$v = c\left[\frac{(1+v_0/c)e^{-g(t-t_0)/c} - (1-v_0/c)e^{g(t-t_0)/c}}{(1+v_0/c)e^{-g(t-t_0)/c} + (1-v_0/c)e^{g(t-t_0)/c}}\right] \tag{B9}$$

for the position and velocity of the particle at time t.

In the general-relativistic framework, the gravitational field outside the uniform sphere is described by the Schwarzschild metric [4] in terms of the Schwarzschild coordinates (ct, r, θ, ϕ)

$$ds^2 = c^2 d\tau^2 = \left(1-\frac{r_s}{r}\right)c^2 dt^2 - \frac{dr^2}{\left(1-\frac{r_s}{r}\right)} - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{B10}$$

where ds is the interval between neighboring events, τ is the proper time, and $r_s = 2GM/c^2$ is the Schwarzschild radius. For purely radial motion [3,5] along the line ϕ =constant in the equatorial plane $\theta = \pi/2$, the metric Eq. (B10) is simplified, with $d\phi = d\theta = 0$, to

$$ds^2 = c^2 d\tau^2 = \left(1 - \frac{r_s}{r}\right) c^2 dt^2 - \frac{dr^2}{(1 - r_s / r)} \quad \text{(B11)}$$

and the geodesic equations are reduced to

$$\frac{cd^2 t}{c^2 d\tau^2} + \left(\frac{r_s / r^2}{1 - r_s / r}\right) \frac{cdt}{cd\tau} \frac{dr}{cd\tau} = 0\,, \quad \text{(B12)}$$

$$\frac{d^2 r}{c^2 d\tau^2} + \left(1 - \frac{r_s}{r}\right)\left(\frac{r_s}{2r^2}\right)\left(\frac{cdt}{cd\tau}\right)^2 - \left(1 - \frac{r_s}{r}\right)^{-1}\left(\frac{r_s}{2r^2}\right)\left(\frac{dr}{cd\tau}\right)^2 = 0\,. \quad \text{(B13)}$$

The local velocity [4,6] of the particle, measured by a local observer who is at rest at a particular Schwarzschild radial coordinate and is next to the particle, is

$$v = \left(1 - \frac{r_s}{r}\right)^{-1} \frac{dr}{dt} = \left(1 - \frac{2GM}{c^2 r}\right)^{-1} \frac{dr}{dt}\,. \quad \text{(B14)}$$

The integral of Eq. (B12), which is given by

$$\frac{cdt}{cd\tau} = k\left(1 - \frac{r_s}{r}\right)^{-1}, \quad \text{(B15)}$$

where k is a constant, and the integral of Eq. (B13), which is given by Eq. (B11), together with the initial condition $v = v_0$ at $r = r_0$, lead to the following expression for dr/dt:

$$\left(\frac{dr}{dt}\right)^2 = \left(1 - \frac{2GM}{c^2 r}\right)^2 \left(1 - \frac{2GM}{c^2 r_0}\right)^{-1} \left[v_0^2 \left(1 - \frac{2GM}{c^2 r}\right) + 2GM\left(\frac{1}{r} - \frac{1}{r_0}\right)\right]. \quad \text{(B16)}$$

If $|r - r_0|/r_0 \ll 1$ and $r_0 \approx R$, substituting Eqs. (B2) and (B3) into Eq. (B16) and integrating it with initial condition $r = r_0$ at $t = t_0$ yields the general-relativistic position of the particle at time t

$$r - r_0 = -\frac{c^2}{2g}\left(1 - \frac{2gr_0}{c^2}\right)\left\{1 - \left\{\frac{1}{2}\left[\left(1 + \frac{v_0}{c}\right)e^{-\frac{g(t-t_0)}{c}} + \left(1 - \frac{v_0}{c}\right)e^{\frac{g(t-t_0)}{c}}\right]\right\}^{-2}\right\}. \quad \text{(B17)}$$

In the limit of weak gravity ($2gr/c^2 \ll 1$ and $2gr_0/c^2 \ll 1$), Eq. (B17) reduces to the special-relativistic Eq. (B8). In the limit of weak gravity and low speed ($v/c \ll 1$, $v_0/c \ll 1$ and $g(t - t_0)/c \ll 1$), Eq. (B17) reduces to the Newtonian Eq. (B5).

Substituting Eqs. (B14), (B2), (B3) and (B17) sequentially into Eq. (B16) yields the general-relativistic velocity of the particle at time *t*, which is the same as the special-relativistic Eq. (B9). In the limit of low speed, Eq. (B9) reduces to the Newtonian Eq. (B6).

References

1. Lapidus IR (1972) The falling body problem in general relativity. Am. J. Phys. 40: 1509-1510.
2. Lapidus IR (1972) Motion of a relativistic particle acted upon by a constant force and a uniform gravitational field. Am. J. Phys. 40: 984-988.
3. Srinivasa Rao KN (1966) The motion of a falling particle in a Schwarzschild field. Ann. Inst. Henri Poincare, Sect. A 5: 227-233.
4. Landau LD, Lifshitz EM (1975) The classical theory of fields. Oxford: Pergamon Press.
5. Srinivasa Rao KN, Gopala Rao AV (1974) Falling body in the theories of

gravitation. J. Phys. A 7**:** 485-488.

6. Zel'dovich YaB, Novikov ID (1996) Relativistic astrophysics vol. 1: Stars and relativity. New York: Dover Publications.